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Existence of nontrivial nonnegative periodic solutions for a class of doubly degenerate parabolic equation with nonlocal terms ☆

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Abstract

In this paper, the authors establish the existence of nontrivial nonnegative periodic solutions for a class of doubly degenerate parabolic equation with nonlocal terms by using the theory of Leray–Schauder's degree.

Keywords: Doubly degenerate; Parabolic; Periodic; Leray–Schauder's degree

1. Introduction

In this paper, we consider the following boundary value problem for periodic doubly degenerate parabolic equation with nonlocal terms

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(\left|\nabla u^m\right|^{p-2}\nabla u^m\right) = \left(a - \Phi[u]\right)u, \qquad (x,t) \in Q_T, \tag{1.1}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u(x,T), \qquad x \in \Omega, \tag{1.3}$$

where $m \geqslant 1$, $p \geqslant 2$, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $Q_T = \Omega \times (0,T)$, $\Phi[u]: L^2(\Omega)^+ \to \mathbb{R}^+$ is a bounded continuous functional, $L^2(\Omega)^+ = \{u \in L^2(\Omega) \mid u \geqslant 0,$

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a.e. in $\Omega\}$, $\mathbb{R}^+ = [0, +\infty)$. This problem models some interesting phenomena in mathematical biology, where $u = u(x,t)$ represents the density of the species at position x and time t, $a = a(x,t)$ represents the maximal value of the natural increasing rate at location x and time t, and thus $a - \Phi[u]$ denotes the actual increasing rate. In this model, the rate of actual increase is influenced not only by the density of the species at some local point but also by the amount of the total species, which is described by the nonlocal term $\Phi[u]$. The homogeneous Dirichlet boundary value condition (1.2) describes that the boundary we consider in this model is lethal to the species.

Our consideration in this paper is motivated by the model proposed by W. Allegretto and P. Nistri [2], who studied the equation

$$\frac{\partial u}{\partial t} - \Delta u = f\big(x, t, \Phi[u], u, a\big)u, \tag{1.4}$$

and obtained the existence of nontrivial nonnegative periodic solutions. Here the typical case of $f(x, t, \Phi[u], u, a)$ is $a - \Phi[u]$. Rui Huang, Yifu Wang and Yuanyuan Ke [7] extend the results to a class of degenerate parabolic equations with nonlocal terms

$$\frac{\partial u}{\partial t} - \Delta u^m = \big(a - \Phi[u]\big)u,$$

which is just the case of our equation for $p = 2$. During the recent years, many authors have focused their eyes on the problems of semilinear and quasilinear equations with nonlocal terms, see, for example, [2,4,5,10,16]. While, due to the relevant connections to gas or fluid flows media and population dynamics, periodic problems for quasilinear degenerate parabolic equations have been the subject of extensive study, see [1,3,8,9,11–14] and references therein.

In this paper, we consider a class of doubly degenerate diffusion equation—the well-known non-Newtonian polytropic filtration equation, which is used to describe the nonstationary flow in a porous medium of fluids with a power dependence of the tangential stress on the velocity of the displacement under polytropic conditions; it has been intensively studied (see [14,15] and references therein). Equation (1.1) is degenerate when $u = 0$, or when the gradient of u vanishes. This degenerate equation exhibiting a doubly nonlinearity generalizes the porous medium equation ($p = 2$) and the parabolic p-Laplace equation ($m = 1$). If $p = 2$, $m = 1$, then Eq. (1.1) becomes a nondegenerate parabolic equation and heat equation is its special case. In our paper, due to the nonlinear term being of logistic type, we mainly emphasize finding the uniform lower bound of maximum modulus of the related solutions, which is much more difficult than finding the upper bound estimates in some circumstances. In fact, here the result of [6] on the sup-bounds estimate of $|\nabla u|$ plays an important role in getting the uniform lower bound.

This paper is organized as follows. In Section 2, we introduce some necessary preliminaries and give the statement of our main result. In Section 3, we give the proof of Theorem 2.1. More precisely, we obtain the existence of nontrivial nonnegative solutions of problem (1.1)–(1.3) by first using the method of Leray–Schauder degree to get the nontrivial nonnegative solution $u_{\varepsilon\eta}$ to the corresponding regularized periodic problem, which has a lower bound independent of ε, η, and then passing to a limit by first letting $\eta \to 0$ and then $\varepsilon \to 0$.

2. Preliminaries and the statement of the main result

Assume that $\Phi[\cdot]$ and $a(x,t)$ satisfy the following conditions:

(A1) $\Phi[\cdot]: L^2(\Omega)^+ \to \mathbb{R}^+$ is a bounded continuous functional satisfying

$$C_1\|u\|^2_{L^2(\Omega)} \leqslant \Phi[u] \leqslant C_2\|u\|^2_{L^2(\Omega)},$$

where $0 < C_1 \leqslant C_2$ are constants independent of u, $\mathbb{R}^+ = [0, +\infty)$, $L^2(\Omega)^+ = \{u \in L^2(\Omega) \mid u \geqslant 0, \text{ a.e. in } \Omega\}$.

(A2) $a(x,t) \in C_T(\bar{Q}_T)$ may change sign, but

$$\left\{x \in \Omega\colon \frac{1}{T}\int_0^T a(x,t)\,dt > 0\right\} \neq \emptyset,$$

where $C_T(\bar{Q}_T)$ is a class of functions which are continuous in $\bar{\Omega} \times \mathbb{R}$ and T-periodic with respect to t.

By (A2) and the continuity of function $a(x,t)$, there exist $x_0 \in \Omega$, $r_0 > 0$ and constant $a_0 > 0$ such that $\frac{1}{T}\int_0^T a(x,t)\,dt \geqslant a_0$ for all $x \in B(x_0, r_0) \subset \Omega$.

Let μ_1 be the first eigenvalue of the following eigenvalue problem

$$\begin{aligned} -\Delta v &= \mu v \quad \text{in } B\left(x_0, \frac{1}{2}r_0\right), \\ v &= 0 \quad \text{on } \partial B\left(x_0, \frac{1}{2}r_0\right). \end{aligned}$$

Our main efforts center on the discussion of generalized solutions, since the regularity follows from a quite standard approach. Hence we give the following definition of generalized solutions of the problem (1.1)–(1.3).

Definition 1. A function u is said to be a generalized solution of the problem (1.1)–(1.3), if $u \in L^\infty(Q_T) \cap C_T(\bar{Q}_T)$, $u^m \in L^p(0,T; W_0^{1,p}(\Omega))$ and u satisfies

$$\iint_{Q_T} \left(-u\frac{\partial\varphi}{\partial t} + \left|\nabla u^m\right|^{p-2}\nabla u^m \nabla\varphi - \big(a - \Phi[u]\big)u\varphi\right)dx\,dt = 0, \tag{2.5}$$

for any $\varphi \in C^1(\bar{Q}_T)$ with $\varphi(x,0) = \varphi(x,T)$ and $\varphi|_{\partial\Omega\times(0,T)} = 0$.

Due to the degeneracy of Eq. (1.1), we should consider the following regularized problem

$$\frac{\partial u_{\varepsilon\eta}}{\partial t} - \operatorname{div}\Big(\big(\left|A(u_{\varepsilon\eta})\nabla u_{\varepsilon\eta}\right|^2 + \eta\big)^{\frac{p-2}{2}} A(u_{\varepsilon\eta})\nabla u_{\varepsilon\eta}\Big) = \big(a - \Phi[u_{\varepsilon\eta}]\big)u_{\varepsilon\eta}, \quad (x,t) \in Q_T, \tag{2.6}$$

$$u_{\varepsilon\eta}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{2.7}$$

$$u_{\varepsilon\eta}(x,0) = u_{\varepsilon\eta}(x,T), \quad x \in \Omega, \tag{2.8}$$

where $A(u_{\varepsilon\eta}) = mu_{\varepsilon\eta}^{m-1} + \varepsilon$, ε is a constant satisfying $0 < \varepsilon \leqslant 1/2$ and η is a constant satisfying $0 < \eta \leqslant 1/2$. The desired solution of the problem (1.1)–(1.3) will be obtained as a limit of the solutions $u_{\varepsilon\eta}$ of the problem (2.6)–(2.8).

For fixed $\varepsilon, \eta > 0$, Eq. (2.6) is uniformly parabolic. However, though $u_{\varepsilon\eta}$ is smooth enough, we cannot ensure $\Phi[u_{\varepsilon\eta}]$ is smoother than C^0. So we would not expect the right side of (2.6) to have C^α smoothness. Furthermore, we do not expect the problem (2.6)–(2.8) to have a classical solution. Now we should define the strong generalized solution of the problem (2.6)–(2.8).

Definition 2. A function $u_{\varepsilon\eta}$ is called a strong generalized solution of problem (2.6)–(2.8), if $u_{\varepsilon\eta} \in L^\infty(Q_T) \cap C_T(\bar{Q}_T) \cap \overset{\bullet}{W}{}_q^{2,1}(Q_T)$ and $u_{\varepsilon\eta}$ satisfies Eq. (2.6) almost everywhere.

In order to employ topological methods to deal with the existence of nontrivial nonnegative solutions of the problem (2.6)–(2.8), we introduce a map by considering the following problem

$$\frac{\partial u_{\varepsilon\eta}}{\partial t} - \operatorname{div}\Big(\big(\tau^\beta \big|B(u_{\varepsilon\eta})\nabla u_{\varepsilon\eta}\big|^2 + \eta\big)^{\frac{p-2}{2}} B(u_{\varepsilon\eta})\nabla u_{\varepsilon\eta}\Big) = f, \quad (x,t) \in Q_T, \tag{2.9}$$

$$u_{\varepsilon\eta}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{2.10}$$

$$u_{\varepsilon\eta}(x,0) = u_{\varepsilon\eta}(x,T), \quad x \in \Omega, \tag{2.11}$$

where $\beta = \frac{2}{m-1}$, $B(u_{\varepsilon\eta}) = \tau m u_{\varepsilon\eta}^{m-1} + \varepsilon$, $\tau \in [0,1]$ is a given parameter. Then, for any given $\tau \in [0,1]$, $f \in C_T(\bar{Q}_T)$, we define a map $u_{\varepsilon\eta} = G(\tau, f)$ with $G : [0,1] \times C_T(\bar{Q}_T) \to C_T(\bar{Q}_T)$. Similarly to the discussion in [12], we can infer that the map $u_{\varepsilon\eta} = G(\tau, f)$ is a compact continuous map. Let $f(v) = (a - \Phi[v])v^+$, where $v^+ = \max\{v, 0\}$. It is not difficult to see that if a nonnegative function $u_{\varepsilon\eta}$ solves $u_{\varepsilon\eta} = G(1, f(u_{\varepsilon\eta}))$, then it is also a nonnegative solution of the problem (2.6)–(2.8). Hence the existence of the nonnegative solution of the problem (2.6)–(2.8) is equivalent to the existence of the nonnegative fixed point of the map $u_{\varepsilon\eta} = G(1, f(v))$.

Our main result is the following.

Theorem 1. *If the assumptions* (A1), (A2) *hold, then the problem* (1.1)–(1.3) *admits a nontrivial nonnegative periodic solution u.*

3. Proof of the main result

First, we prove the nonnegativity of the solutions to the regularized problem. For the sake of simplicity, we will denote $u_{\varepsilon\eta}$ by u in the proof of the following lemmas and corollaries.

Lemma 1. *If a nontrivial function $u_{\varepsilon\eta} \in C_T(\bar{Q}_T)$ solves $u_{\varepsilon\eta} = T(\tau, \sigma(a - \Phi[u_{\varepsilon\eta}])u_{\varepsilon\eta}^+ + (1-\tau))$, $\tau \in [0,1]$, $\sigma \in [0,1]$, then*

$$u_{\varepsilon\eta}(x,t) > 0, \quad \textit{for all } (x,t) \in Q_T.$$

And hence this nonnegative function u solves $u_{\varepsilon\eta} = T(\tau, \sigma(a - \Phi[u_{\varepsilon\eta}])u_{\varepsilon\eta} + (1-\tau))$.

Proof. We first prove that $u \geqslant 0$. Multiplying Eq. (2.9) by u^- and integrating the resulting relation over Q_T, we have

$$\begin{aligned}
&\int_\Omega \int_0^T \sigma\big(a - \Phi[u]\big)u^+u^-\,dt\,dx + \int_\Omega \int_0^T (1-\tau)u^-\,dt\,dx \\
&\quad = \int_\Omega \int_0^T \frac{\partial u}{\partial t}u^-\,dt\,dx + \int_\Omega \int_0^T \big(\tau^\beta\big|B(u)\nabla u\big|^2 + \eta\big)^{\frac{p-2}{2}}\big(\tau m u^{m-1} + \varepsilon\big)\nabla u \nabla u^-\,dt\,dx \\
&\quad = \frac{1}{2}\int_\Omega \int_0^T \frac{\partial}{\partial t}(u^-)^2\,dt\,dx + \varepsilon \int_\Omega \int_0^T \big(\tau^\beta\big|B(u)\nabla u\big|^2 + \eta\big)^{\frac{p-2}{2}}|\nabla u^-|^2\,dt\,dx \\
&\qquad + \frac{4m\tau}{(m+1)^2}\int_\Omega \int_0^T \big(\tau^\beta\big|B(u)\nabla u\big|^2 + \eta\big)^{\frac{p-2}{2}}\big|\nabla (u^-)^{\frac{m+1}{2}}\big|^2\,dt\,dx,
\end{aligned}$$

where $u^- = \min\{u, 0\}$. The first term of the right-hand side in the above integral equality vanishes due to the periodicity of u, and first term of the left-hand side also obviously equals 0. Furthermore, the second term of the left-hand side is nonpositive. Hence noticing that $(\tau^\beta |B(u)\nabla u|^2 + \eta)^{\frac{p-2}{2}} > 0$, we can see that

$$\int_\Omega \int_0^T |\nabla u^-|^2 \, dt \, dx \leqslant 0.$$

The Poincaré inequality gives

$$\int_\Omega |u^-|^2 \, dx \leqslant C \int_\Omega |\nabla u^-|^2 \, dx,$$

where $C > 0$ is a constant depending only on Ω. Then we have

$$\int_0^T \int_\Omega |u^-|^2 \, dx \, dt \leqslant C \int_0^T \int_\Omega |\nabla u^-|^2 \, dx \, dt \leqslant 0,$$

which, together with the continuity of u^-, implies that $u^- = 0$, $\forall (x,t) \in Q_T$. Then $u = u^+ \geqslant 0$ for $(x,t) \in Q_T$.

Next we prove $u > 0$. Since u is nontrivial, there exist $x \in \Omega$ and $\xi \in (0, T]$ such that $u(x, \xi) \not\equiv 0$. We choose a nontrivial function $0 \leqslant \psi(x) \in C_0^\infty(\Omega)$ with $\psi(x) < u(x, \xi)$. Suppose K is a positive constant and v solves the following problem

$$\begin{aligned}
&\frac{\partial v}{\partial t} - \operatorname{div}\Big(\big(\tau^\beta \big|B(v)\nabla v\big|^2 + \eta\big)^{\frac{p-2}{2}} B(v)\nabla v\Big) + Kv = 0, && x \in \Omega,\ t > \xi,\\
&v(x,t) = 0, && (x,t) \in \partial\Omega \times [\xi, T],\\
&v(x,\xi) = \psi(x), && x \in \Omega.
\end{aligned}$$

Noticing that $u \in C_T(\bar{Q}_T)$ and the assumption (A1), we have $\Phi[u] \in C_T(\bar{Q}_T)$. This, together with $a \in C_T(\bar{Q}_T)$ gives $a - \Phi[u] \in C_T(\bar{Q}_T)$. By the comparison theorem, we have $u(x,t) \geqslant v(x,t)$ for a large enough constant K. By the maximum principle, we have $v(x,t) > 0$ for any given $x \in \Omega$ and $t > \xi$. Noticing that u is T-periodic with respect to t, we have $u(x,t) \geqslant v(x,t) > 0$ for all $t > 0$, namely $u(x,t) > 0$, $\forall (x,t) \in Q_T$. The proof is complete. □

Next, we should verify that the map $u = G(\tau, f(\cdot))$ satisfies some necessary conditions which allow us to use the homotopy invariance of the Leray–Schauder degree.

Lemma 2. *If a nontrivial function $u_{\varepsilon\eta}$ solves $u_{\varepsilon\eta} = G(1, \sigma f(u_{\varepsilon\eta}))$, $\sigma \in [0,1]$, then there exists a positive constant R independent of η, ε and σ, such that*

$$\|u_{\varepsilon\eta}\|_{L^\infty(Q_T)} < R. \tag{3.12}$$

Proof. Suppose u is a nontrivial solution of $u = G(1, \sigma f(u))$, $\sigma \in [0,1]$. By Lemma 1, we can see that u also solves $u = G(1, \sigma(a - \Phi[u])u)$. Multiplying Eq. (2.9) by u as $\tau = 1$ and integrating the resulting relation over Q_T, we have

$$\iint\limits_{Q_T} u\frac{\partial u}{\partial t}\,dt\,dx + \iint\limits_{Q_T} \left(\left|A(u)\nabla u\right|^2+\eta\right)^{\frac{p-2}{2}} A(u)|\nabla u|^2\,dt\,dx$$
$$= \iint\limits_{Q_T} \sigma\bigl(a-\Phi[u]\bigr)u^2\,dt\,dx.$$

Due to the periodicity of u with respect to t, we have

$$\iint\limits_{Q_T} u\frac{\partial u}{\partial t}\,dt\,dx = 0,$$

which, together with

$$\iint\limits_{Q_T} \left(\left|A(u)\nabla u\right|^2+\eta\right)^{\frac{p-2}{2}} A(u)|\nabla u|^2\,dt\,dx \geqslant 0 \quad \text{and} \quad \sigma\in[0,1],$$

implies that

$$\iint\limits_{Q_T} \bigl(a-\Phi[u]\bigr)u^2\,dt\,dx \geqslant 0.$$

Let $M=\sup_{(x,t)\in\bar{Q}_T} a(x,t)$. By the assumption (A1), we can see that

$$\begin{aligned}
0 &\leqslant \iint\limits_{Q_T} \bigl(a(x,t)-\Phi[u]\bigr)u^2\,dt\,dx \\
&\leqslant \iint\limits_{Q_T} Mu^2\,dt\,dx - \iint\limits_{Q_T} \bigl(u^2\Phi[u]\bigr)\,dx\,dt \\
&\leqslant M\iint\limits_{Q_T} u^2\,dt\,dx - C_1\int_0^T\biggl(\int\limits_\Omega u^2\,dx\biggr)^2 dt,
\end{aligned}$$

namely,

$$\int_0^T\biggl(\int\limits_\Omega u^2\,dx\biggr)^2 dt \leqslant C\int_0^T\biggl(\int\limits_\Omega u^2\,dx\biggr)dt, \tag{3.13}$$

where C is a constant independent of σ, ε and η. The Cauchy inequality gives that

$$\int\limits_\Omega u^2\,dx \leqslant \frac{1}{4\alpha^2}+\alpha^2\biggl(\int\limits_\Omega u^2\,dx\biggr)^2$$

holds for any constant $\alpha>0$, which implies

$$\int_0^T\int\limits_\Omega u^2\,dx\,dt \leqslant \frac{T}{4\alpha^2}+\alpha^2\int_0^T\biggl(\int\limits_\Omega u^2\,dx\biggr)^2 dt. \tag{3.14}$$

Choosing a suitably small α and combining (3.14) with (3.13), we have

$$\|u\|_{L^2(Q_T)} \leqslant C, \tag{3.15}$$

where C is a constant independent of σ, ε and η. Since the continuity of $a(x,t)$ and the assumption (A1), there exists a constant $K > 0$, such that u satisfies

$$\frac{\partial u}{\partial t} - \operatorname{div}\Big(\big(\big|A(u)\nabla u\big|^2 + \eta\big)^{\frac{p-2}{2}} A(u)\nabla u\Big) \leqslant Ku. \tag{3.16}$$

By the results in Ref. [6, Theorem 3.2, p. 121], we have

$$\sup_{(x,t)\in\Omega\times[T/2,3T/2]} u(x,t) \leqslant C \int_{T/2}^{3T/2}\int_{\Omega} u\,dx\,dt.$$

It follows from the periodicity of u and the Hölder inequality that

$$\|u\|_{L^\infty(Q_T)} \leqslant C\|u\|_{L^2(Q_T)},$$

where C is a constant independent of η, ε and σ. Combining the above inequality with (3.15), we obtain

$$\|u\|_{L^\infty(Q_T)} \leqslant C,$$

where C is a constant independent of η, ε and σ. Choosing a constant $R > C$, we can see that the lemma holds and the proof is complete. □

Corollary 1. *There exists a positive constant R such that*

$$\deg\big(I - G\big(1, f(\cdot)\big), B_R, 0\big) = 1,$$

where B_R is a ball centered at the origin with radius R in $L^\infty(Q_T)$.

Proof. It follows from Lemma 2 that there exists a positive constant R independent of σ, such that

$$u \neq G\big(1, \sigma f(u)\big), \quad \forall u \in \partial B_R,\ \sigma \in [0,1].$$

Hence the degree is well defined on B_R. From the homotopy invariance of the Leray–Schauder degree, we can see that

$$\deg\big(I - G\big(1, f(\cdot)\big), B_R, 0\big) = \deg\big(I - G\big(1, \sigma f(\cdot)\big), B_R, 0\big), \quad \forall \sigma \in [0,1]. \tag{3.17}$$

Obviously, $G(1,0) = 0$. Then, letting $\sigma = 0$ in (3.17) yields

$$\deg\big(I - G\big(1, f(\cdot)\big), B_R, 0\big) = \deg(I, B_R, 0) = 1.$$

The proof is complete. □

Lemma 3. *There exists a constant $r > 0$, such that no nontrivial solutions $u_{\varepsilon\eta}$ of the equation $u_{\varepsilon\eta} = G(\tau, f(u_{\varepsilon\eta}) + (1-\tau))$, $\tau \in [0,1]$, satisfy*

$$0 < \|u_{\varepsilon\eta}\|_{L^\infty(Q_T)} \leqslant r.$$

Proof. Suppose $u = G(\tau, f(u) + (1-\tau))$, $\tau \in [0,1]$ admits a nontrivial solution u satisfying $0 < \|u\|_{L^\infty(Q_T)} \leqslant r$. By Lemma 1, u also solves $u = G(\tau, (a - \Phi[u])u + (1-\tau))$ and $u(x,t) > 0$. Hence for any given $\phi(x) \in C_0^\infty(\Omega)$, we can choose $\frac{\phi^2}{u}$ as a test function. Multiplying Eq. (2.9) by $\frac{\phi^2}{u}$ and integrating the resulting relation over Q_T, we obtain

$$\iint_{Q_T} \frac{\phi^2}{u}\frac{\partial u}{\partial t}\,dt\,dx + \iint_{Q_T} \left(\tau^\beta \left|B(u)\nabla u\right|^2 + \eta\right)^{\frac{p-2}{2}} B(u)\nabla u \nabla\left(\frac{\phi^2}{u}\right) dt\,dx$$
$$= \iint_{Q_T} \left(\phi^2\left(a - \Phi[u]\right) + (1-\tau)\frac{\phi^2}{u}\right) dt\,dx. \tag{3.18}$$

By the periodicity of u, the first term of the left-hand side in (3.18) satisfies

$$\iint_{Q_T} \frac{\phi^2}{u}\frac{\partial u}{\partial t}\,dt\,dx = \int_\Omega \phi^2 \int_0^T \frac{\partial(\ln u)}{\partial t}\,dt\,dx = 0. \tag{3.19}$$

The second term of the left-hand side in (3.18) can be rewritten as

$$\iint_{Q_T} \left(\tau^\beta \left|B(u)\nabla u\right|^2 + \eta\right)^{\frac{p-2}{2}} B(u)\nabla u \nabla\left(\frac{\phi^2}{u}\right) dt\,dx$$
$$= \iint_{Q_T} \left(\tau^\beta \left|B(u)\nabla u\right|^2 + \eta\right)^{\frac{p-2}{2}} B(u)\nabla u \left(\frac{\phi}{u}\nabla\phi + \phi\nabla\left(\frac{\phi}{u}\right)\right) dt\,dx$$
$$= \iint_{Q_T} \left(\tau^\beta \left|B(u)\nabla u\right|^2 + \eta\right)^{\frac{p-2}{2}} B(u)\left(\frac{\nabla\phi}{u}\right)\left(u\nabla\phi - u^2\nabla\left(\frac{\phi}{u}\right)\right) dt\,dx$$
$$+ \iint_{Q_T} \left(\tau^\beta \left|B(u)\nabla u\right|^2 + \eta\right)^{\frac{p-2}{2}} B(u)\phi\nabla u \nabla\left(\frac{\phi}{u}\right) dt\,dx$$
$$= \iint_{Q_T} \left(\tau^\beta \left|B(u)\nabla u\right|^2 + \eta\right)^{\frac{p-2}{2}} B(u)|\nabla\phi|^2\,dt\,dx$$
$$- \iint_{Q_T} \left(\tau^\beta \left|B(u)\nabla u\right|^2 + \eta\right)^{\frac{p-2}{2}} B(u)\left(u\nabla\phi - \phi\nabla u\right)\nabla\left(\frac{\phi}{u}\right) dt\,dx$$
$$= \iint_{Q_T} \left(\tau^\beta \left|B(u)\nabla u\right|^2 + \eta\right)^{\frac{p-2}{2}} B(u)|\nabla\phi|^2\,dt\,dx$$
$$- \iint_{Q_T} \left(\tau^\beta \left|B(u)\nabla u\right|^2 + \eta\right)^{\frac{p-2}{2}} B(u)u^2\left|\nabla\left(\frac{\phi}{u}\right)\right|^2 dt\,dx.$$

So the inequality

$$\iint_{Q_T} \left(\tau^\beta \left|B(u)\nabla u\right|^2 + \eta\right)^{\frac{p-2}{2}} B(u)|\nabla\phi|^2\,dt\,dx - \iint_{Q_T} \phi^2\left(a - \Phi[u]\right) dt\,dx$$
$$= \iint_{Q_T} (1-\tau)\frac{\phi^2}{u}\,dt\,dx + \iint_{Q_T} \left(\tau^\beta \left|B(u)\nabla u\right|^2 + \eta\right)^{\frac{p-2}{2}} B(u)u^2\left|\nabla\left(\frac{\phi}{u}\right)\right|^2 dt\,dx \geqslant 0$$

follows.

By Theorem 5.1 and some remarks in [6, pp. 238, 243], it follows that there exists a constant $\gamma = \gamma(N, p)$ such that

$$\sup_{[(x_0,t_0)+Q(\frac{1}{2}r_0,\frac{1}{2}\rho)]} \left|A(u)\nabla u\right|$$
$$= C(N, p, r_0, a_0, \mu_1)\bigg(\iint\limits_{[(x_0,t_0)+Q(r_0,\rho)]} \left|A(u)\nabla u\right|^p \,dt\,dx\bigg)^{\frac{1}{2}} \wedge \frac{1}{2}\left(\frac{a_0}{4\mu_1}\right)^{\frac{1}{p-2}},$$

for any $(x_0, t_0) \in Q_{(T,3T)} = \Omega \times (T, 3T)$, $[(x_0, t_0) + Q(r_0, \rho)] \subset Q_{(T,3T)}$ and $\rho = \min\{T, \frac{\sqrt{a_0 r_0}}{2^{\frac{p+6}{2}}}\}$. On the other hand, by (2.6)–(2.8), we have

$$\iint\limits_{Q_T} \left|A(u)\nabla u\right|^p \,dt\,dx \leqslant \max_{Q_T}\left|a(x,t)\right| \iint\limits_{Q_T} \left(|u|^{m+1} + |u|^2\right) dt\,dx.$$

So

$$\sup_{[(x_0,t_0)+Q(\frac{1}{2}r_0,\frac{1}{2}\rho)]} \left|A(u)\nabla u\right|$$
$$\leqslant C(N, p, r_0, a_0, \mu_1)\bigg(\iint\limits_{Q_T} \left(|u|^{m+1} + |u|^2\right) dt\,dx\bigg)^{\frac{1}{2}} \wedge \frac{1}{2}\left(\frac{a_0}{4\mu_1}\right)^{\frac{1}{p-2}},$$

which implies

$$\left\|A(u)\nabla u\right\|_{L^\infty(B(x_0,r_0)\times(0,T))} \leqslant C\left(\|u\|^{\frac{m+1}{2}}_{L^\infty(Q_T)} + \|u\|_{L^\infty(Q_T)}\right) \wedge \frac{1}{2}\left(\frac{a_0}{4\mu_1}\right)^{\frac{1}{p-2}},$$

where C is a constant independent of ε, η and τ.

Since

$$\left(\tau^\beta\left|B(u)\nabla u\right|^2 + \eta\right)^{\frac{p-2}{2}} \leqslant 2^{\frac{p-2}{2}}\left(\left|A(u)\nabla u\right|^{p-2} + \eta^{\frac{p-2}{2}}\right),$$

we have

$$\iint\limits_{Q_T} 2^{\frac{p-2}{2}}\left(\left|A(u)\nabla u\right|^{p-2} + \eta^{\frac{p-2}{2}}\right)B(u)|\nabla\phi|^2 \,dt\,dx - \iint\limits_{Q_T} \phi^2\left(a - \Phi[u]\right) dt\,dx$$
$$\geqslant \iint\limits_{Q_T} \left(\tau^\beta\left|B(u)\nabla u\right|^2 + \eta\right)^{\frac{p-2}{2}} B(u)|\nabla\phi|^2 \,dt\,dx - \iint\limits_{Q_T} \phi^2\left(a - \Phi[u]\right) dt\,dx \geqslant 0.$$

From $\varepsilon, \eta \in (0, \frac{1}{2})$, $\tau \in (0, 1)$, we have $B(u) = \tau m u^{m-1} + \varepsilon \leqslant m r^{m-1} + \frac{1}{2}$. By the approximating process, we can let $\phi = \phi_1$, where ϕ_1 is the positive eigenfunction of the first eigenvalue μ_1. Then we have

$$\iint\limits_{B(x_0,\frac{1}{2}r_0)\times(0,T)} \phi_1^2\left(a - \Phi[u]\right) dt\,dx$$
$$\leqslant \iint\limits_{B(x_0,\frac{1}{2}r_0)\times(0,T)} 2^{\frac{p-2}{2}}\left(\left|A(u)\nabla u\right|^{p-2} + \eta^{\frac{p-2}{2}}\right)B(u)|\nabla\phi|^2 \,dt\,dx$$

$$\leqslant \iint\limits_{B(x_0,\frac{1}{2}r_0)\times(0,T)} \left(C\left(r^{\frac{m+1}{2}}+r\right)^{p-2} \wedge \frac{a_0}{4\mu_1} + (2\eta)^{\frac{p-2}{2}}\right)\left(mr^{m-1}+\frac{1}{2}\right)|\nabla\phi|^2\,dt\,dx$$

$$= T\left(C\mu_1\left(r^{\frac{m+1}{2}}+r\right)^{p-2} \wedge \frac{a_0}{4} + 2^{\frac{p-2}{2}}\mu_1\eta^{\frac{p-2}{2}}\right)\left(mr^{m-1}+\frac{1}{2}\right) \int\limits_{B(x_0,\frac{1}{2}r_0)} \phi_1^2\,dx.$$

On the other hand,

$$\iint\limits_{B(x_0,\frac{1}{2}r_0)\times(0,T)} \phi_1^2\left(a-\Phi[u]\right)dt\,dx$$

$$\geqslant \int\limits_{B(x_0,\frac{1}{2}r_0)} \phi_1^2(x)\left(\int_0^T a(x,t)\,dt - C_2|\Omega|r^2T\right)dx$$

$$\geqslant \left(T\left(a_0 - C_2|\Omega|r^2\right)\right) \int\limits_{B(x_0,\frac{1}{2}r_0)} \phi_1^2(x)\,dx,$$

where $|\Omega|$ denotes the Lebesgue measure of the domain Ω.

Therefore we obtain

$$a_0 \leqslant C_2|\Omega|r^2 + \left(C\mu_1\left(r^{\frac{m+1}{2}}+r\right)^{p-2} \wedge \frac{a_0}{4} + 2^{\frac{p-2}{2}}\mu_1\eta^{\frac{p-2}{2}}\right)\left(mr^{m-1}+\frac{1}{2}\right).$$

Obviously if we let

$$\eta \leqslant \frac{1}{2}\left(\frac{a_0}{4\mu_1}\right)^{\frac{2}{p-2}}, \qquad r \leqslant \min\left\{\sqrt[m-1]{1/(2m)}, \left(\frac{a_0}{4C_2}\right)^{\frac{1}{2}}, \frac{1}{2}\left(\frac{a_0}{4C\mu_1}\right)^{\frac{1}{p-2}}, 1\right\},$$

we can get

$$a_0 \leqslant \frac{a_0}{4} + \left(\frac{a_0}{4} \wedge \frac{a_0}{4} + \frac{a_0}{4}\right) = \frac{3a_0}{4}.$$

This inequality does not hold. Therefore there exists one positive constant $r > 0$, such that no nontrivial solutions u of the equation $u = G(\tau, f(u) + (1-\tau))$, $\tau \in [0,1]$, satisfy

$$0 < \|u\|_{L^\infty(Q_T)} \leqslant r.$$

Thus we complete the proof. □

Corollary 2. *There exists a small positive constant r satisfying $r < R$, such that*

$$\deg\left(I - G\left(1, f(\cdot)\right), B_r, 0\right) = 0,$$

where B_r is a ball centered at the origin with radius r in $L^\infty(Q_T)$.

Proof. It follows from Lemma 3 that there exists a positive constant r independent of τ, such that

$$u \neq G\left(\tau, f(u) + (1-\tau)\right), \quad \forall u \in \partial B_r,\ \tau \in [0,1].$$

Hence the degree is well defined on B_r. From the homotopy invariance of the Leray–Schauder degree, we can see that

$$\deg\big(I - G\big(1, f(\cdot)\big), B_r, 0\big) = \deg\big(I - G\big(0, f(\cdot) + 1\big), B_r, 0\big). \tag{3.20}$$

From Lemma 3, we can infer that $u = G(0, f(u) + 1)$ admits no nontrivial solution in B_r. Obviously, $u = 0$ is not a trivial solution of $u = G(0, f(u) + 1)$. Then $u = G(0, f(u) + 1)$ admits no solution in B_r. Namely, $\deg(I - G(0, f(\cdot) + 1), B_r, 0) = 0$. Then, from (3.20) we have

$$\deg\big(I - G\big(1, f(\cdot)\big), B_r, 0\big) = 0.$$

The proof is complete. □

Theorem 2. *If the assumptions* (A1), (A2) *hold, then the problem* (2.6)–(2.8) *admits a nontrivial nonnegative periodic solution* $u_{\varepsilon\eta} \in L^\infty(Q_T) \cap C_T(\bar{Q}_T)$ *and* $u_{\varepsilon\eta}^m \in L^p(0, T; W_0^{1,p}(\Omega))$.

Proof. From the Corollaries 1 and 2, we can see that

$$\deg\big(I - G\big(1, f(\cdot)\big), \Sigma, 0\big) = 1,$$

where $\Sigma = B_R \setminus B_r$, B_ξ is a ball centered at the origin with radius ξ in $L^\infty(Q_T)$, R and r are positive constants and satisfy $R > r$. By the properties of the Leray–Schauder degree, together with Lemmas 1 and 3, we can infer that the problem (2.6)–(2.8) admits a nontrivial nonnegative periodic solution. The proof is complete. □

Next we show some estimates on $u_{\varepsilon\eta}$.

Lemma 4. $u_{\varepsilon\eta}$ *satisfies the following*:

$$\iint_{Q_T} \Big(\big(\big|\nabla u_{\varepsilon\eta}^m + \varepsilon \nabla u_{\varepsilon\eta}\big|^2 + \eta\big)^{\frac{p-2}{2}}\Big)\big|\nabla u_{\varepsilon\eta}^m + \varepsilon \nabla u_{\varepsilon\eta}\big|^2\,dx\,dt \leqslant C, \tag{3.21}$$

$$\iint_{Q_T} \left|\frac{\partial u_{\varepsilon\eta}^m}{\partial t}\right|^2 dx\,dt \leqslant C, \tag{3.22}$$

$$\iint_{Q_T} \left|\frac{\partial u_{\varepsilon\eta}}{\partial t}\right|^2 dx\,dt \leqslant C. \tag{3.23}$$

Proof. By Lemmas 2 and 3 we know the solution $u_{\varepsilon\eta} \in \Sigma$ of the problem (2.6)–(2.8) satisfies

$$r < \|u_{\varepsilon\eta}\|_{L^\infty(Q_T)} < R, \tag{3.24}$$

where r, R are constants independent of ε, η.

Multiplying Eq. (3.16) by $u_{\varepsilon\eta}^m + \varepsilon u_{\varepsilon\eta}$, and integrating the result over Q_T, we obtain

$$\begin{aligned}
&\iint_{Q_T} \big(u_{\varepsilon\eta}^m + \varepsilon u_{\varepsilon\eta}\big)\frac{\partial u_{\varepsilon\eta}}{\partial t}\,dx\,dt + \iint_{Q_T} \Big(\big(\big|\nabla u_{\varepsilon\eta}^m + \varepsilon \nabla u_{\varepsilon\eta}\big|^2 + \eta\big)^{\frac{p-2}{2}}\Big)\big|\big(\nabla u_{\varepsilon\eta}^m + \varepsilon \nabla u_{\varepsilon\eta}\big)\big|^2\,dx\,dt \\
&\quad \leqslant K \iint_{Q_T} \big(u_{\varepsilon\eta}^{m+1} + \varepsilon u_{\varepsilon\eta}^2\big)\,dx\,dt \leqslant K \iint_{Q_T} \big(u_{\varepsilon\eta}^{m+1} + u_{\varepsilon\eta}^2\big)\,dx\,dt,
\end{aligned}$$

where K is a constant independent of ε and η. Due to the periodicity of $u_{\varepsilon\eta}$ and the L^∞ norm bound of $u_{\varepsilon\eta}$, we have

$$\iint_{Q_T} \left(\left(\left|\nabla u_{\varepsilon\eta}^m + \varepsilon \nabla u_{\varepsilon\eta}\right|^2 + \eta\right)^{\frac{p-2}{2}}\right)\left|\nabla u_{\varepsilon\eta}^m + \varepsilon \nabla u_{\varepsilon\eta}\right|^2 dx\, dt \leqslant C,$$

where C is a constant independent of ε and η.

Multiplying Eq. (2.6) by $\frac{\partial}{\partial t}(u_{\varepsilon\eta}^m + \varepsilon u_{\varepsilon\eta})$, and integrating the result over Ω, we obtain

$$\begin{aligned}
&\int_\Omega \frac{\partial u_{\varepsilon\eta}}{\partial t}\frac{\partial u_{\varepsilon\eta}^m}{\partial t}\, dx + \varepsilon \int_\Omega \left|\frac{\partial u_{\varepsilon\eta}}{\partial t}\right|^2 dx + \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \int_0^{|\nabla u_{\varepsilon\eta}^m + \varepsilon \nabla u_{\varepsilon\eta}|^2} (s+\eta)^{\frac{p-2}{2}}\, ds\, dx \\
&\quad = \int_\Omega \left(a - \Phi[u_{\varepsilon\eta}]\right) u_{\varepsilon\eta} \frac{\partial u_{\varepsilon\eta}^m}{\partial t}\, dx + \varepsilon \int_\Omega \left(a - \Phi[u_{\varepsilon\eta}]\right) u_{\varepsilon\eta} \frac{\partial u_{\varepsilon\eta}}{\partial t}\, dx \\
&\quad = \frac{2m}{m+1}\int_\Omega \left(a - \Phi[u_{\varepsilon\eta}]\right) u_{\varepsilon\eta}^{(m+1)/2} \frac{\partial}{\partial t} u_{\varepsilon\eta}^{(m+1)/2}\, dx + \varepsilon \int_\Omega \left(a - \Phi[u_{\varepsilon\eta}]\right) u_{\varepsilon\eta} \frac{\partial u_{\varepsilon\eta}}{\partial t}\, dx \\
&\quad \leqslant \frac{m}{2}\int_\Omega \left|\left(a - \Phi[u_{\varepsilon\eta}]\right) u_{\varepsilon\eta}^{(m+1)/2}\right|^2 dx + \frac{2m}{(m+1)^2}\int_\Omega \left|\frac{\partial}{\partial t} u_{\varepsilon\eta}^{(m+1)/2}\right|^2 dx \\
&\qquad + \frac{1}{4}\int_\Omega \left|\left(a - \Phi[u_{\varepsilon\eta}]\right) u_{\varepsilon\eta}\right|^2 dx + \varepsilon^2 \int_\Omega \left|\frac{\partial u_{\varepsilon\eta}}{\partial t}\right|^2 dx.
\end{aligned}$$

Noticing that the first term of the left side of the above inequality can be rewritten as

$$\int_\Omega \frac{\partial u_{\varepsilon\eta}}{\partial t}\frac{\partial u_{\varepsilon\eta}^m}{\partial t}\, dx = \frac{4m}{(m+1)^2}\int_\Omega \left|\frac{\partial}{\partial t} u_{\varepsilon\eta}^{(m+1)/2}\right|^2 dx.$$

Then we have

$$\begin{aligned}
&\frac{2m}{(m+1)^2}\int_\Omega \left|\frac{\partial}{\partial t} u_{\varepsilon\eta}^{(m+1)/2}\right|^2 dx + \left(\varepsilon - \varepsilon^2\right)\int_\Omega \left|\frac{\partial u_{\varepsilon\eta}}{\partial t}\right|^2 dx \\
&\qquad + \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \int_0^{|\nabla u_{\varepsilon\eta}^m + \varepsilon \nabla u_{\varepsilon\eta}|^2} (s+\eta)^{\frac{p-2}{2}}\, ds\, dx \\
&\quad \leqslant \frac{m}{2}\int_\Omega \left|\left(a - \Phi[u_{\varepsilon\eta}]\right) u_{\varepsilon\eta}^{(m-1)/2}\right|^2 dx + \frac{1}{4}\int_\Omega \left|\left(a - \Phi[u_{\varepsilon\eta}]\right) u_{\varepsilon\eta}\right|^2 dx,
\end{aligned}$$

which together with the bound of a, $\Phi[u_{\varepsilon\eta}]$, $u_{\varepsilon\eta}$ and the periodicity of $u_{\varepsilon\eta}$ yields

$$\iint_{Q_T} \left|\frac{\partial u_{\varepsilon\eta}^{(m+1)/2}}{\partial t}\right|^2 dx\, dt \leqslant C,$$

where C is a constant independent of ε. Noticing the bound of $u_{\varepsilon\eta}$ again, we have

$$\iint\limits_{Q_T} \left|\frac{\partial u_{\varepsilon\eta}^m}{\partial t}\right|^2 dx\,dt = \frac{4m^2}{(m+1)^2} \iint\limits_{Q_T} u_{\varepsilon\eta}^{m-1} \left|\frac{\partial u_{\varepsilon\eta}^{(m+1)/2}}{\partial t}\right|^2 dx\,dt \leqslant C,$$

$$\iint\limits_{Q_T} \left|\frac{\partial u_{\varepsilon\eta}}{\partial t}\right|^2 dx\,dt \leqslant C,$$

where C is a constant independent of ε and η. The proof is complete. □

Theorem 3. *If the assumptions* (A1), (A2) *hold, then a nontrivial nonnegative periodic solution* $u_\varepsilon \in L^\infty(Q_T) \cap C_T(\bar{Q}_T)$ *and* $u_\varepsilon^m \in L^p(0,T; W_0^{1,p}(\Omega))$ *solves*

$$u_{\varepsilon t} - \operatorname{div}\left(\left|\nabla u_\varepsilon^m + \varepsilon \nabla u_\varepsilon\right|^{p-2}\left(\nabla u_\varepsilon^m + \varepsilon \nabla u_\varepsilon\right)\right) = \left(a - \Phi[u_\varepsilon]\right)u_\varepsilon. \tag{3.25}$$

Proof. By (3.21), (3.22) and (3.24), we see that for any given $\varepsilon > 0$ there is a subsequence of $\{u_{\varepsilon\eta}\}$ (without loss of generality, we may denote it by $\{u_{\varepsilon\eta}\}$) and a periodic function u_ε such that as $\eta \to 0$,

$$u_{\varepsilon\eta} \to u_\varepsilon \qquad \text{in } L^\infty(Q_T) \cap C(Q_T),$$
$$\frac{\partial u_{\varepsilon\eta}}{\partial t} \rightharpoonup \frac{\partial u_\varepsilon}{\partial t} \qquad \text{in } L^2(Q_T).$$

Noticing (3.21), we also get

$$\iint\limits_{Q_T} \left|\left(\left|\nabla u_{\varepsilon\eta}^m + \varepsilon \nabla u_{\varepsilon\eta}\right|^2 + \eta\right)^{\frac{p-2}{2}} \frac{\partial (u_{\varepsilon\eta}^m + \varepsilon u_{\varepsilon\eta})}{\partial x_i}\right|^{\frac{p}{p-1}} dx\,dt \leqslant C. \tag{3.26}$$

This implies that there exists $\mu_{\varepsilon i} \in L^{\frac{p}{p-1}}(Q_T)$, $i = 1, \dots, n$, such that

$$\left(\left|\nabla u_{\varepsilon\eta}^m + \varepsilon \nabla u_{\varepsilon\eta}\right|^2 + \eta\right)^{\frac{p-2}{2}} \frac{\partial (u_{\varepsilon\eta}^m + \varepsilon u_{\varepsilon\eta})}{\partial x_i} \rightharpoonup \mu_{\varepsilon i} \quad \text{in } L^{\frac{p}{p-1}}(Q_T). \tag{3.27}$$

Hence,

$$\iint\limits_{Q_T} \left(u_\varepsilon \varphi_t - \mu_\varepsilon \nabla\varphi + \left(a - \Phi[u_\varepsilon]\right)u_\varepsilon \varphi\right) dx\,dt = 0, \tag{3.28}$$

where $\mu_\varepsilon = (\mu_{\varepsilon 1}, \dots, \mu_{\varepsilon n})$, $\varphi \in C^1(\bar{Q}_T)$ with $\varphi|_{\partial\Omega \times (0,T)} = 0$ and $\varphi(x,0) = \varphi(x,T)$.

Now for any φ given as before, we show

$$\iint\limits_{Q_T} \left|\nabla u_\varepsilon^m + \varepsilon \nabla u_\varepsilon\right|^{p-2}\left(\nabla u_\varepsilon^m + \varepsilon \nabla u_\varepsilon\right)\nabla\varphi\,dx\,dt = \iint\limits_{Q_T} \mu_\varepsilon \nabla\varphi\,dx\,dt. \tag{3.29}$$

Noticing the fact that for the vector function $F(X) = |X|^{p-2}X$,

$$F'(X) = |X|^{p-2}I + (p-2)|X|^{p-4}XX^\top$$

is a nonnegative matrix, therefore,

$$\left(F(X) - F(Y)\right) \cdot (X - Y) \geqslant 0. \tag{3.30}$$

Then for any $v \in L^p(0,T; W_0^{1,p}(\Omega))$, $\zeta \in C^1(\bar{Q}_T)$, $0 \leqslant \zeta \leqslant 1$, with $\zeta|_{\partial\Omega \times (0,T)} = 0$ and $\zeta(x,0) = \zeta(x,T)$,

$$\iint_{Q_T} \zeta\left(\left|\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\right|^{p-2}\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)-|\nabla v|^{p-2}\nabla v\right) \cdot\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}-v\right)dx\,dt\geqslant 0. \tag{3.31}$$

Noticing that

$$\begin{aligned}
&\iint_{Q_T} \zeta\left(\left|\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\right|^2+\eta\right)^{(p-2)/2}\left|\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\right|^2dx\,dt\\
&\quad=\iint_{Q_T}\zeta\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\left(a-\Phi[u_{\varepsilon\eta}]\right)u_{\varepsilon\eta}\,dx\,dt+\iint_{Q_T}\left(\frac{1}{m+1}u_{\varepsilon\eta}^{m+1}+\frac{\varepsilon}{2}u_{\varepsilon\eta}^2\right)\zeta_t\,dx\,dt\\
&\qquad-\iint_{Q_T}\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\left(\left|\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\right|^2+\eta\right)^{(p-2)/2}\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\nabla\zeta\,dx\,dt,
\end{aligned}$$

and for $p\geqslant 2$,

$$\left(\left|\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\right|^2+\eta\right)^{(p-2)/2}\left|\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\right|^2\geqslant\left|\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\right|^p,$$

combing with (3.31), we get

$$\begin{aligned}
&\iint_{Q_T}\zeta\left(u_{\varepsilon\eta}^{m+1}+\varepsilon u_{\varepsilon\eta}^2\right)\left(a-\Phi[u_{\varepsilon\eta}]\right)dx\,dt+\iint_{Q_T}\left(\frac{1}{m+1}u_{\varepsilon\eta}^{m+1}+\frac{\varepsilon}{2}u_{\varepsilon\eta}^2\right)\zeta_t\,dx\,dt\\
&\quad-\iint_{Q_T}\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\left(\left|\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\right|^2+\eta\right)^{(p-2)/2}\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\nabla\zeta\,dx\,dt\\
&\quad-\iint_{Q_T}\zeta\left|\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\right|^{p-2}\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\nabla v\,dx\,dt\\
&\quad-\iint_{Q_T}\zeta|\nabla v|^{p-2}\nabla v\cdot\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}-v\right)dx\,dt\geqslant 0.
\end{aligned}\tag{3.32}$$

Since

$$\begin{aligned}
&\left(\left|\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\right|^2+\eta\right)^{(p-2)/2}\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\\
&\quad=\left|\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\right|^{p-2}\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\\
&\qquad+\frac{(p-2)\eta}{2}\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\int_0^1\left(\left|\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\right|^2+\eta s\right)^{(p-4)/2}ds
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\eta\to 0}\iint_{Q_T}\frac{(p-2)\eta}{2}\int_0^1\left(\left|\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\right|^2+\eta s\right)^{(p-4)/2}ds\\
&\quad\times\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\nabla\left(u_{\varepsilon\eta}^m+\varepsilon u_{\varepsilon\eta}\right)\nabla\zeta\,dx\,dt=0,
\end{aligned}$$

letting $\eta\to 0$ in (3.32), then

$$\iint_{Q_T} \zeta\big(u_\varepsilon^{m+1} + \varepsilon u_\varepsilon^2\big)\big(a - \Phi[u_\varepsilon]\big)\,dx\,dt + \iint_{Q_T} \left(\frac{1}{m+1}u_\varepsilon^{m+1} + \frac{\varepsilon}{2}u_\varepsilon^2\right)\zeta_t\,dx\,dt \\ - \iint_{Q_T} \big(u_\varepsilon^m + \varepsilon u_\varepsilon\big)\mu_\varepsilon \nabla\zeta\,dx\,dt - \iint_{Q_T} \zeta\mu_\varepsilon\nabla v\,dx\,dt \\ - \iint_{Q_T} \zeta|\nabla v|^{p-2}\nabla v\nabla\big(u_\varepsilon^m + \varepsilon u_\varepsilon - v\big)\,dx\,dt \geqslant 0. \tag{3.33}$$

Setting $\varphi = \zeta(u_\varepsilon^m + \varepsilon u_\varepsilon)$ in (3.28), we obtain

$$\iint_{Q_T} \zeta\big(u_\varepsilon^{m+1} + \varepsilon u_\varepsilon^2\big)\big(a - \Phi[u_\varepsilon]\big)\,dx\,dt + \iint_{Q_T} \left(\frac{1}{m+1}u_\varepsilon^{m+1} + \frac{\varepsilon}{2}u_\varepsilon^2\right)\zeta_t\,dx\,dt \\ - \iint_{Q_T} \big(u_\varepsilon^m + \varepsilon u_\varepsilon\big)\mu_\varepsilon \nabla\zeta\,dx\,dt = \iint_{Q_T} \zeta\mu_\varepsilon\nabla\big(u_\varepsilon^m + \varepsilon u_\varepsilon\big)\,dx\,dt,$$

substituting the above equation into (3.33), we get

$$\iint_{Q_T} \zeta\big(\mu_\varepsilon - |\nabla v|^{p-2}\nabla v\big)\nabla\big(u_\varepsilon^m + \varepsilon u_\varepsilon - v\big)\,dx\,dt \geqslant 0. \tag{3.34}$$

By taking $v = u_\varepsilon^m + \varepsilon u_\varepsilon - \delta\varphi$, $\delta \geqslant 0$, in (3.34), we have

$$\iint_{Q_T} \zeta\big(\mu_\varepsilon - \big|\nabla\big(u_\varepsilon^m + \varepsilon u_\varepsilon - \delta\varphi\big)\big|^{p-2}\nabla\big(u_\varepsilon^m + \varepsilon u_\varepsilon - \delta\varphi\big)\big)\nabla\varphi\,dx\,dt \geqslant 0,$$

where $\varphi \in C^1(\bar{Q}_T)$ with $\varphi(x,0) = \varphi(x,T)$ and $\varphi|_{\partial\Omega\times(0,T)} = 0$. Then, letting $\delta \to 0$, we see that

$$\iint_{Q_T} \zeta\big(\mu_\varepsilon - \big|\nabla u_\varepsilon^m + \varepsilon\nabla u_\varepsilon\big|^{p-2}\big(\nabla u_\varepsilon^m + \varepsilon\nabla u_\varepsilon\big)\big)\nabla\varphi\,dx\,dt \geqslant 0.$$

Obviously, if we let $\delta \leqslant 0$, we can get the inverted inequality. So we can obtain (3.29) by choosing suitable ζ, s.t. $\operatorname{supp}\varphi \subset \operatorname{supp}\zeta$ and $\zeta = 1$ on $\operatorname{supp}\varphi$. Therefore, we have

$$\iint_{Q_T} \big(u_\varepsilon\varphi_t - \big|\nabla u_\varepsilon^m + \varepsilon\nabla u_\varepsilon\big|^{p-2}\big(\nabla u_\varepsilon^m + \varepsilon\nabla u_\varepsilon\big)\nabla\varphi + \big(a - \Phi[u_\varepsilon]\big)u_\varepsilon\varphi\big)\,dx\,dt = 0, \tag{3.35}$$

that is u_ε is a solution of Eq. (3.25). The proof is complete. □

Now we prove the main result of this paper.

Proof of Theorem 1. By Lemma 4 and Theorem 3, we know that $u_\varepsilon \in L^\infty(Q_T) \cap C_T(\bar{Q}_T)$, $u_\varepsilon^m \in L^p(0,T;W_0^{1,p}(\Omega))$. Hence by (3.21), (3.23), (3.24) and (3.36) we see that there is a subsequence of $\{u_\varepsilon\}$ (without loss of generality, we denote it by $\{u_\varepsilon\}$) and a periodic function u such that as $\varepsilon \to 0$,

$$\begin{aligned} &u_\varepsilon \to u && \text{in } L^\infty(Q_T) \cap C(Q_T), \\ &\frac{\partial u_\varepsilon}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} && \text{in } L^2(Q_T). \end{aligned}$$

By (3.24), we know that

$$r < \|u_\varepsilon\|_{L^\infty(Q_T)} < R,$$

where r, R are constants independent of ε. Setting $\varphi = -(u_\varepsilon^m + \varepsilon u_\varepsilon)$ in (3.35), we have

$$\iint\limits_{Q_T} -u_\varepsilon\big(u_\varepsilon^m + \varepsilon u_\varepsilon\big)_t\,dx\,dt + \iint\limits_{Q_T} \big|\nabla u_\varepsilon^m + \varepsilon\nabla u_\varepsilon\big|^p\,dx\,dt = \iint\limits_{Q_T} \big(a - \Phi[u_\varepsilon]\big)\big(u_\varepsilon^{m+1} + u_\varepsilon^2\big)\,dx\,dt.$$

Noticing that $\iint_{Q_T} -u_\varepsilon(u_\varepsilon^m + \varepsilon u_\varepsilon)_t = 0$ due to the periodicity of u_ε and combing with the above inequality, we get

$$\iint\limits_{Q_T} \big|\nabla u_\varepsilon^m + \varepsilon\nabla u_\varepsilon\big|^p\,dx\,dt \leqslant C. \tag{3.36}$$

This implies that there exists $\nu_i \in L^{\frac{p}{p-1}}(Q_T)$, $i = 1, \dots, n$, such that

$$\big|\nabla u_\varepsilon^m + \varepsilon\nabla u_\varepsilon\big|^{p-2}\frac{\partial(u_\varepsilon^m + \varepsilon u_\varepsilon)}{\partial x_i} \rightharpoonup \nu_i \quad \text{in } L^{\frac{p}{p-1}}(Q_T).$$

Hence,

$$\iint\limits_{Q_T} \big(u\varphi_t - \nu\nabla\varphi + \big(a - \Phi[u]\big)u\varphi\big)\,dx\,dt = 0, \tag{3.37}$$

where $\nu = (\nu_1, \dots, \nu_n)$, $\varphi \in C^1(\bar{Q}_T)$ with $\varphi(x,0) = \varphi(x,T)$ and $\varphi|_{\partial\Omega\times(0,T)} = 0$.

Now for any φ given as before, we show

$$\iint\limits_{Q_T} \big|\nabla u^m\big|^{p-2}\nabla u^m\nabla\varphi\,dx\,dt = \iint\limits_{Q_T} \nu\nabla\varphi\,dx\,dt. \tag{3.38}$$

Using (3.30), we see that for any $v \in L^p(0,T;W_0^{1,p}(\Omega))$, $\zeta \in C^1(\bar{Q}_T)$, $0 \leqslant \zeta \leqslant 1$, with $\zeta|_{\partial\Omega\times(0,T)} = 0$ and $\zeta(x,0) = \zeta(x,T)$,

$$\iint\limits_{Q_T} \big(\big|\nabla\big(u_\varepsilon^m + \varepsilon u_\varepsilon\big)\big|^{p-2}\nabla\big(u_\varepsilon^m + \varepsilon u_\varepsilon\big) - |\nabla v|^{p-2}\nabla v\big)\cdot\nabla\big(u_\varepsilon^m + \varepsilon u_\varepsilon - v\big)\,dx\,dt \geqslant 0. \tag{3.39}$$

Noticing that

$$\begin{aligned}
&\iint\limits_{Q_T} \zeta\big(\big|\nabla\big(u_\varepsilon^m + \varepsilon u_\varepsilon\big)\big|^p\big)\,dx\,dt \\
&\quad = \iint\limits_{Q_T} \zeta\big(u_\varepsilon^m + \varepsilon u_\varepsilon\big)\big(a - \Phi[u_\varepsilon]\big)u_\varepsilon\,dx\,dt - \iint\limits_{Q_T} \left(\frac{1}{m+1}u_\varepsilon^{m+1} + \frac{\varepsilon}{2}u_\varepsilon^2\right)\zeta_t\,dx\,dt \\
&\qquad - \iint\limits_{Q_T} \big(u_\varepsilon^m + \varepsilon u_\varepsilon\big)\big|\nabla\big(u_\varepsilon^m + \varepsilon u_\varepsilon\big)\big|^{p-2}\nabla\big(u_\varepsilon^m + \varepsilon u_\varepsilon\big)\nabla\zeta\,dx\,dt,
\end{aligned}$$

combing with (3.39), we get

$$\iint_{Q_T} \zeta\big(u_\varepsilon^{m+1} + \varepsilon u_\varepsilon^2\big)\big(a - \Phi[u_\varepsilon]\big)\,dx\,dt - \iint_{Q_T} \bigg(\frac{1}{m+1}u_\varepsilon^{m+1} + \frac{\varepsilon}{2}u_\varepsilon^2\bigg)\zeta_t\,dx\,dt$$
$$- \iint_{Q_T} \big(u_\varepsilon^m + \varepsilon u_\varepsilon\big)\big|\nabla\big(u_\varepsilon^m + \varepsilon u_\varepsilon\big)\big|^{p-2}\nabla\big(u_\varepsilon^m + \varepsilon u_\varepsilon\big)\nabla\zeta\,dx\,dt$$
$$- \iint_{Q_T} \zeta\big|\nabla\big(u_\varepsilon^m + \varepsilon u_\varepsilon\big)\big|^{p-2}\nabla\big(u_\varepsilon^m + \varepsilon u_\varepsilon\big)\nabla v\,dx\,dt$$
$$- \iint_{Q_T} \zeta|\nabla v|^{p-2}\nabla v\nabla\big(u_\varepsilon^m + \varepsilon u_\varepsilon - v\big)\,dx\,dt \geqslant 0. \tag{3.40}$$

Letting $\varepsilon \to 0$ in (3.40), then

$$\iint_{Q_T} \zeta u^{m+1}\big(a - \Phi[u]\big)\,dx\,dt - \iint_{Q_T} \frac{1}{m+1}u^{m+1}\zeta_t\,dx\,dt$$
$$- \iint_{Q_T} u^m \nu\nabla\zeta\,dx\,dt - \iint_{Q_T} \zeta\nu\nabla v\,dx\,dt$$
$$- \iint_{Q_T} \zeta|\nabla v|^{p-2}\nabla v\nabla\big(u^m - v\big)\,dx\,dt \geqslant 0. \tag{3.41}$$

Setting $\varphi = \zeta u^m$ in (3.37), we obtain

$$\iint_{Q_T} \zeta u^{m+1}\big(a - \Phi[u]\big)\,dx\,dt - \iint_{Q_T} \frac{1}{m+1}u^{m+1}\zeta_t\,dx\,dt$$
$$= \iint_{Q_T} u^m \nu\nabla\zeta\,dx\,dt + \iint_{Q_T} \zeta\nu\nabla u^m\,dx\,dt,$$

substituting the above equation into (3.41), we get

$$\iint_{Q_T} \zeta\big(\nu - |\nabla v|^{p-2}\nabla v\big)\nabla\big(u^m - v\big)\,dx\,dt \geqslant 0. \tag{3.42}$$

By taking $v = u^m - \delta\varphi$, $\delta \geqslant 0$, in (3.42) and then letting $\delta \to 0$, we have

$$\iint_{Q_T} \zeta\big(\nu - \big|\nabla u^m\big|^{p-2}\nabla u^m\big)\nabla\varphi\,dx\,dt \geqslant 0,$$

where $\varphi \in C^1(\bar{Q}_T)$ with $\varphi(x,0) = \varphi(x,T)$ and $\varphi|_{\partial\Omega\times(0,T)} = 0$. Obviously, if we let $\delta \leqslant 0$, we can get the inverted inequality. So we can obtain (3.38) by choosing the same ζ as in Theorem 3.

Therefore, we have

$$\iint_{Q_T} \big(u\varphi_t - \big|\nabla u^m\big|^{p-2}\nabla u^m\nabla\varphi + \big(a - \Phi[u]\big)u\varphi\big)\,dx\,dt = 0,$$

that is we obtain a periodic solution u of the problem (1.1)–(1.3). Furthermore, by Lemmas 1 and 3, we can see that this solution u is nontrivial and nonnegative. The proof is complete. □

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References

[1] N. Alaa, M. Iguernane, Weak periodic solutions of some quasilinear parabolic equations with data measures, J. Inequal. Pure Appl. Math. 3 (3) (2002) 46.
[2] W. Allegretto, P. Nistri, Existence and optimal control for periodic parabolic equations with nonlocal term, IMA J. Math. Control Inform. 16 (1) (1999) 43–58.
[3] M. Badii, Periodic solutions for a class of degenerate evolution problem, Nonlinear Anal. 44 (2001) 499–508.
[4] A. Calsina, C. Perello, Equations for biological evolution, Proc. Roy. Soc. Edinburgh Sect. A 125 (1995) 939–958.
[5] A. Calsina, C. Perello, J. Saldana, Non-local reaction–diffusion equations modelling predator–prey coevolution, Publ. Mat. 32 (1994) 315–325.
[6] E. Dibenedetto, Degenerate Parabolic Equations, Springer-Verlag, New York, 1993.
[7] Rui Huang, Yifu Wang, Yuanyuan Ke, Existence of the nontrivial nonnegative periodic solutions for a class of degenerate parabolic equations with nonlocal terms, Discrete Contin. Dyn. Syst. Ser. B 5 (4) (2005) 1005–1014.
[8] G.M. Lieberman, Time-periodic solutions of quasilinear parabolic differential equations I. Dirichlet boundary conditions, J. Math. Anal. Appl. 264 (2) (2001) 617–638.
[9] Zhenhai Liu, Periodic solutions for double degenerate quasilinear parabolic equations, Nonlinear Anal. 51 (7) (2002) 1245–1257.
[10] Pierre Rouchon, Universal bounds for global solutions of a diffusion equation with a nonlocal reaction term, J. Differential Equations 193 (1) (2003) 75–94.
[11] Yifu Wang, Zhuoqun Wu, Jingxue Yin, Time periodic solutions of a class of degenerate parabolic equations, Acta Math. Appl. Sin. Engl. Ser. 16 (2) (2000) 180–187.
[12] Yifu Wang, Jingxue Yin, Zhuoqun Wu, Periodic solutions of evolution p-Laplacian equations with nonlinear sources, J. Math. Anal. Appl. 219 (1) (1998) 76–96.
[13] Yifu Wang, Jingxue Yin, Zhuoqun Wu, Periodic solutions of porous medium equations with weakly nonlinear sources, Northeast. Math. J. 16 (4) (2000) 475–483.
[14] Z.Q. Wu, Q.N. Zhao, J.X. Yin, H.L. Li, Nonlinear Diffusion Equations, World Scientific, Singapore, 2001.
[15] J.N. Zhao, H.J. Yuan, The Cauchy problem of some doubly nonlinear degenerate parabolic equations, Chinese Ann. Math. Ser. A 16 (2) (1995) 179–194 (in Chinese).
[16] Qian Zhou, Yuanyuan Ke, Yifu Wang, Jingxue Yin, Periodic p-Laplacian with nonlocal terms, Comput. Math. Appl. 50 (8–9) (2005) 1293–1302.